(8)

$L_K(E)$ is $\mathbb{Z}$-graded ring (and grading is super nice!)

$$L_K(E) = \frac{K\langle u, e, e^* \mid u \in E^0,\ e \in E^1\rangle}{\begin{array}{l} uv = \delta_{u,v}u \\ s(e)e = e\,r(e) = e \quad \text{and for } e^* \\ \sum_{e \in s^{-1}(v)} ee^* = v \\ e^*f = \delta_{e,f}\, r(e) \quad e,f \in E^1 \end{array}}$$

Now Consider $w: E^1 \longrightarrow \Gamma$ Γ abelian group

and $w(e^*) := -w(e)$, $w(u) = 0$ $\forall u \in E^0$

the free ring $K\langle u, e, e^*\rangle$ is Γ-graded and ideal

is homogeneous $\rightarrow$ $L_K(E)$ is Γ-graded.

Canonical Case $w(e) = 1$ $\forall e \in E^1$ and $\Gamma = \mathbb{Z}$.

Def a ring R is called von-Neuman regular if for any $x \in R$, $\exists\, y \in R$ s.t. $x = xyx$ ($\Longleftrightarrow x \in xRx$)

Ex $M_n(K)$ are von-Neuman regular

(Ken Goodearl's book von Neuman regular ring)

th $L_K(E)$ is von Neuman regular iff E is acyclic

~~[illegible]~~ a acyclic graph $\xrightarrow{\text{roughly}} M_n(K) \times M_m(K)$

Consider $L_K(E)$ as $\mathbb{Z}$-graded ring

th $L_K(E)$ is graded regular ring

($x \in L_K(E)$ homogeneous $\Rightarrow x = xyx$)

towards classification

A unital ring

$\mathcal{V}(A) = \{ [P] \mid P \text{ f.g proj modules} / A \}$

$\mathcal{V}(A)$ is a monoid $\quad [A]+[B] := [A \oplus B]$

Ex $A = K$ field

$\mathcal{V}(A) = \{ [P] \mid P \text{ f.g proj} / A \} \xrightarrow{\text{iso}} \mathbb{N}$

$[K^n] \longmapsto n$

Def The Grothendieck grp $K_0(A) := \mathcal{V}(A)^+$ the group completion

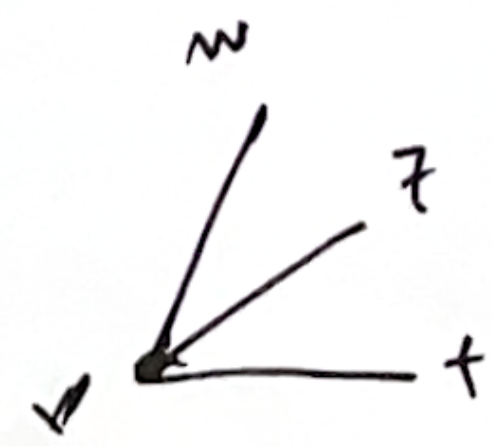
Ex $K_0(K) = \mathbb{Z}$

The K_0 is developed by J. P. Serre, A. Grothendieck, H. Bass

<u>theorem</u> (Ara, Moreno, Pardo) K field, E graph

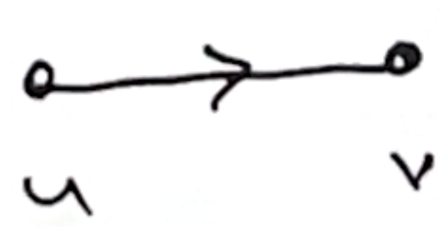
$\mathcal{V}(L_K(E)) \cong M_E$ where

$$M_E = \frac{\langle v \in E^0 \rangle}{\langle v = \sum_{v \xrightarrow{e} w} w \rangle}$$



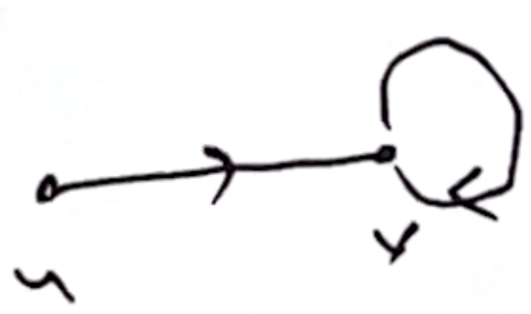
$v = w + z + t$

<u>Ex</u> E



$$M_E = \frac{\langle u, v \rangle}{\langle u = v \rangle} = \mathbb{N}$$

F



$$M_F = \frac{\langle u, v \rangle}{\left\langle \begin{matrix} u = v \\ v = v \end{matrix} \right\rangle} = \mathbb{N}$$

G

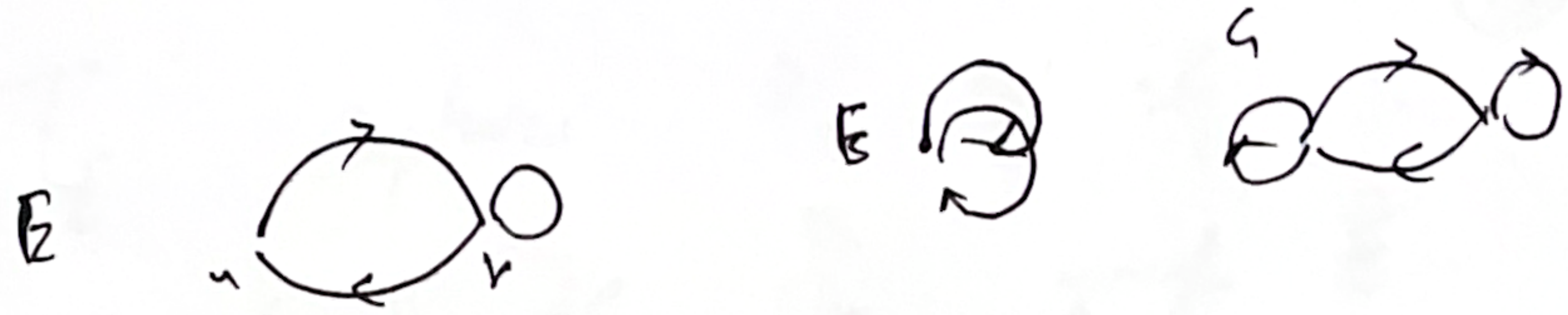
$$M_G = \frac{\langle u, v \rangle}{\left\langle \begin{matrix} u = v \\ v = u \end{matrix} \right\rangle} = \mathbb{N}$$

$L_K(E) = M_2(K)$

$L_K(F) = M_2(K[x, x^{-1}])$

$L_K(E) \neq L_K(F)$

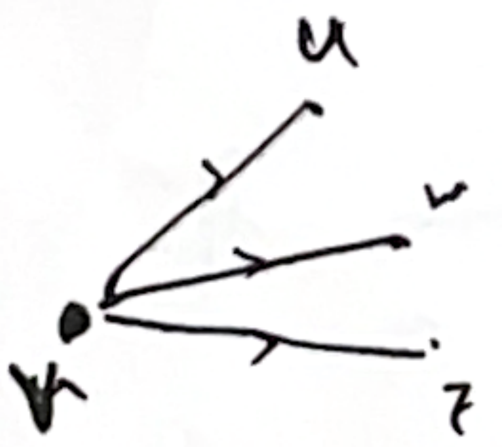
Exercise



prove $M_E \cong M_F \cong M_G$.

talented monoid of a graph

introducing "time evolution" to graph monoid M_E



in M_E, $v = u + w + z$

in T_E, $v = u(1) + w(1) + z(1)$

Def E graph

$$T_E = \frac{\langle v(i) \mid v \in E^0, i \in \mathbb{Z} \rangle}{\langle v(i) = \sum_{v \xrightarrow{e} w} w(i+1) \rangle}$$

T_E is a $\mathbb{Z}$-monoid i.e. $\mathbb{Z} \curvearrowright T_E$

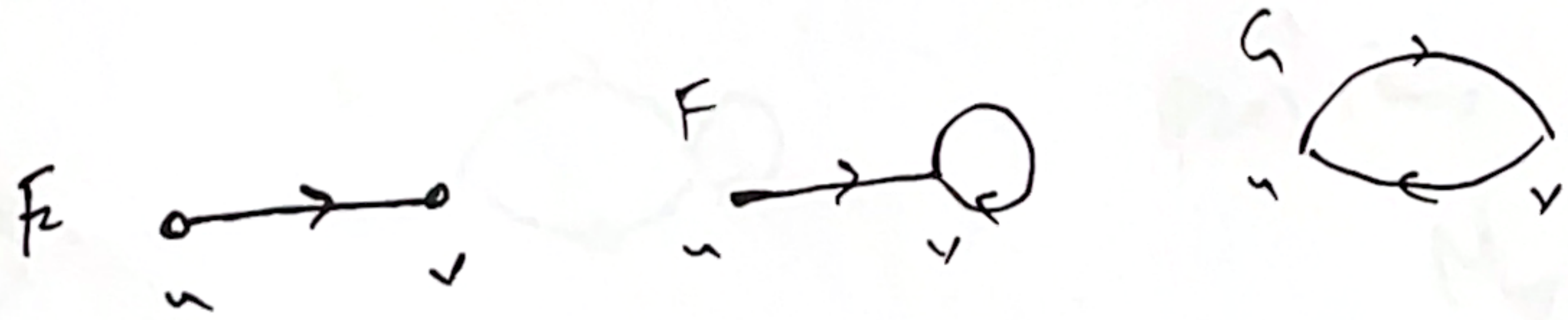
$${}^n v(i) := v(i+n).$$

<u>Claim</u> T_E captures "a lot of geometry" of the graph E.

<u>Theorem</u>

$L(E)$ has property P $\longleftrightarrow$ E has property Q $\longleftrightarrow$ T_E has property R

<u>Ex</u>



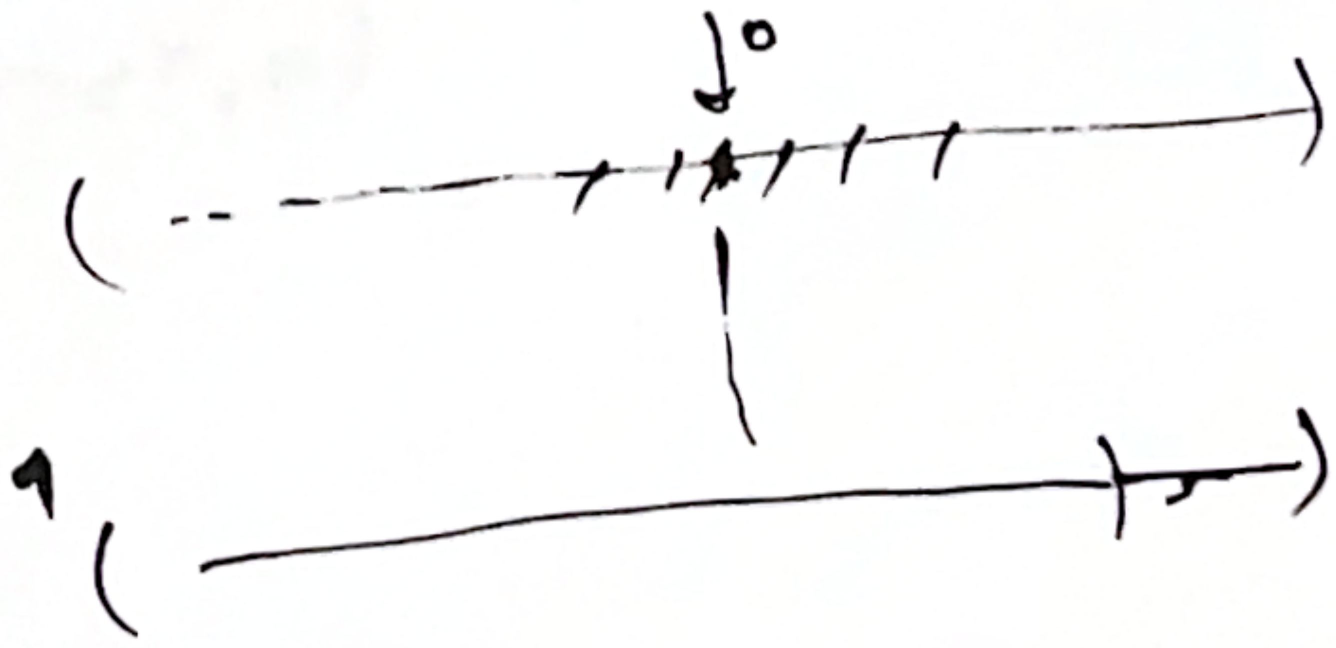
$M_E \cong M_F \cong M_G$

$$M_E = \frac{\langle u(i), v(i) \mid i \in \mathbb{Z} \rangle}{\langle u(i) = v(i+1) \mid i \in \mathbb{Z} \rangle} = \langle \dots, u(-1), u(0), u(1), u(2), \dots \rangle$$

$$\cong \bigoplus_{\mathbb{Z}} \mathbb{N}$$

${}^1 u(0) = u(1)$

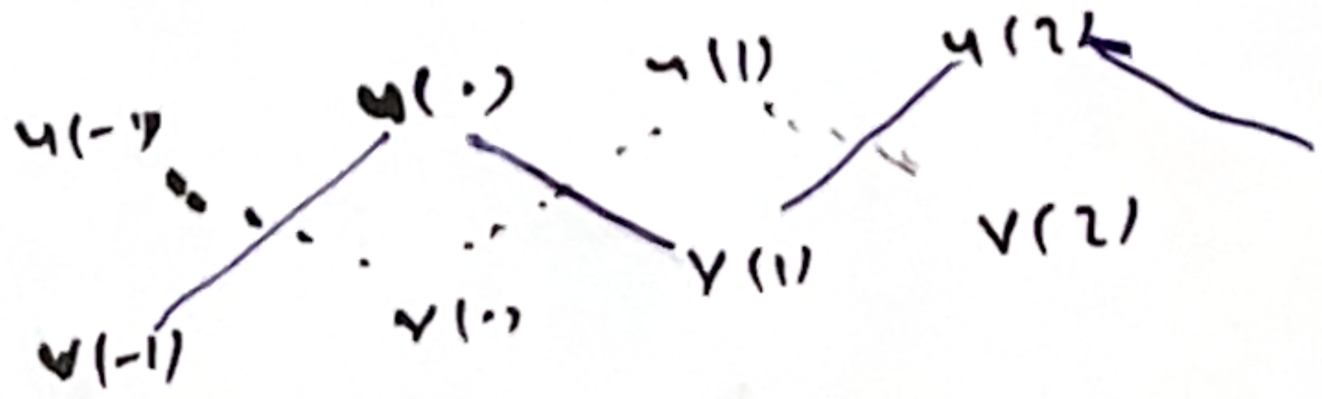
${}^1 (u(0) + u(5)) = u(1) + u(6)$



$\to$ push to the right

$$M_F = \frac{\langle u(i), v(i) \rangle}{\left\langle \begin{matrix} u(i) = v(i+1) \\ v(i) = v(i+1) \end{matrix} \right\rangle} = \mathbb{N}$$

with trivial action



$$M_G = \frac{\langle u(i), v(i) \rangle}{\left\langle \begin{matrix} u(i) = v(i+1) \\ v(i) = u(i+1) \end{matrix} \right\rangle}$$

$\cong \mathbb{N} \oplus \mathbb{N}$

$1_{(a,b)} = (b,a)$

E 

F 

previous exercise

$M_E \cong M_F = \{0, a\}$

	0	a
0	·	a
a	a	a

$$T_E = \mathbb{N}[\tfrac{1}{2}] = \left\{ \begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \\ 1, 2, 3 \ldots \end{matrix} \right\}$$

$$T_F = \left\{ (m,n) \in \mathbb{Z} \oplus \mathbb{Z} \;\middle|\; \frac{1+\sqrt{5}}{2} m + n \geq 0 \right\}$$

$$1_{(m,n)} = (m+n, m)$$

esercizi

$M_E = \{0, a\}$

	0	a
0	0	a
a	a	a

$$T_E = \left\{ (n,m,p) \in \mathbb{Z}^3 \;\middle|\; (n,m,p)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} > 0 \right.$$ when

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\left(-1 + \sqrt[3]{\frac{1}{2}(25 - 3\sqrt{69})} + \sqrt[3]{\frac{1}{2}(25 + 3\sqrt{69})}\right) \\ \frac{1}{3}\left(1 - 5\sqrt[3]{\frac{2}{11 + 3\sqrt{69}}} + \sqrt[3]{\frac{1}{2}(11 + 3\sqrt{69})}\right) \\ 1 \end{pmatrix}$$

$1(n,m,p) := (n+p, n, m+p)$

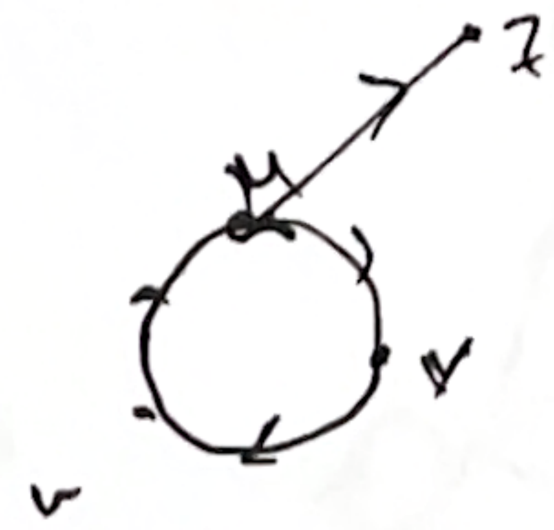
Working with T_E is easy!

Ex

$v \in T_E$

$v = w(1) = u(2) = v(3)$ →

$\boxed{{}^3v = v}$



$v = w(1) = u(2) = v(3) + z(1)$

so ${}^3v < v$

Theorem (Li, ...) E graph, T_E talented monoid

① E has a cycle without exit iff

$\exists\, a \in T_E,\ n \in \mathbb{N}$ s.t. ${}^na = a$

② E has a cycle with exit iff

$\exists\, a \in T_E$ $n \in \mathbb{N}$ s.t. ${}^na > a$

Corollary

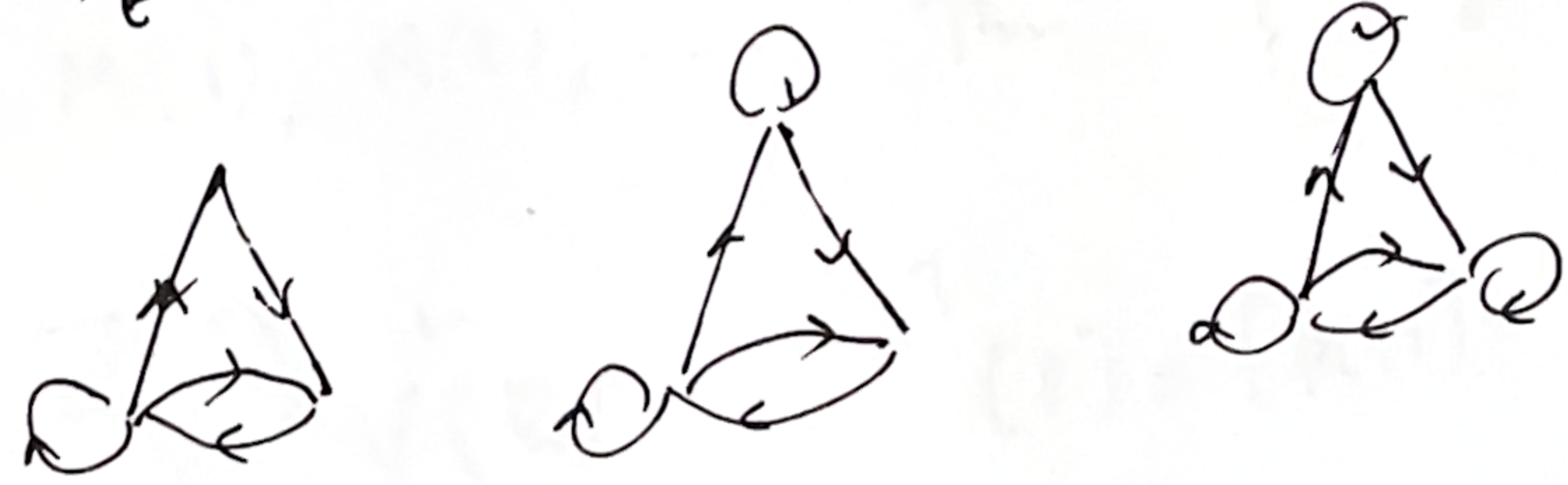
E has condition (L) iff $\mathbb{Z}$ acts freely on T_E

(17)

Research project

T_E can see the number of loops -- how!?

①



② Characterize condition (K)

~~a~~ a vertex on a closed path → is on two closed paths

Connection to Leavitt path algebra

Recall $V(R) = \{ [P] \mid P \text{ f.g proj modules} \}$

↳ R is Γ-graded

$V^{gr}(R) = \{ [P] \mid P \text{ \underline{graded} f.g proj modules} \}$

M is a graded module, there is a notion of suspension (shift) (18)

$M(1), M(2), \ldots$ Then if A is $\mathbb{Z}$-graded

$\mathbb{Z} \curvearrowright \mathcal{V}^{gr}(R)$ $\quad {}^{1}[P] = [P(1)]$

<u>Theorem</u> E graph, $L_k(E)$ Leavitt path alg

$\cancel{K_0^{gr}(L(E)) \cong \mathcal{V}(L(E)) \cong}$

$$\mathcal{V}^{gr}(L(E)) \cong T_E$$

<u>Define</u> graded Grothendieck group $K_0^{gr}(L(E)) \cong \mathcal{V}^{gr}(L(E))^{+}$

<u>Ex</u> $K_0^{gr}(L(8)) = \mathbb{Z}[\tfrac{1}{2}]$

when $K_0(L(8)) = 0$.

Graded classification conjecture

E, F finite graphs

① $L(E) \cong_{gr} L(F)$ iff $\mathcal{T}_E \overset{\mathbb{Z}\curvearrowright \quad \mathbb{Z}\curvearrowright}{\cong} \mathcal{T}_F$

$1_E \longmapsto 1_F$

when $1_E := \sum_{v \in E^0} v$

iff $K_0^{gr}(L(E)) \cong K_0^{gr}(L(F))$ (as $\mathbb{Z}\curvearrowright$)

$[L(E)] \longmapsto [L(F)]$

② ~~$L(E)\cong_{gr}$~~

$Gr\text{-}L(E) \cong Gr\text{-}L(F)$ iff $\mathcal{T}_E \overset{\mathbb{Z}\curvearrowright \quad \mathbb{Z}\curvearrowright}{\cong} \mathcal{T}_F$

category of graded modules

eq

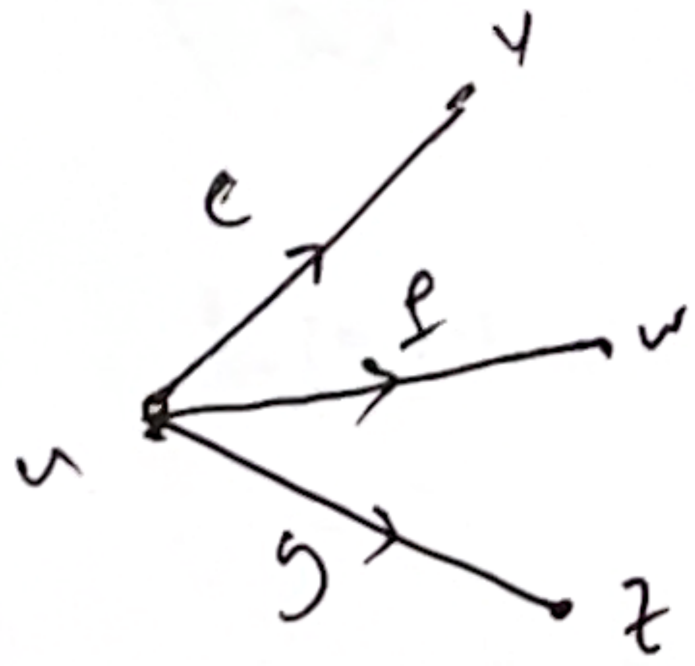
Symbolic dynamics

notion of shift of finite types

Lind-Marcus "Symbolic dynamics"

Weighted Lpa

graph



① $ee^* + ff^* + gg^* = u$

$$(e, f, g)\begin{pmatrix} e^* \\ f^* \\ g^* \end{pmatrix} = u$$

row column

② $e^*e = v$ . .

$e^*f = 0$

$$\begin{pmatrix} e^* \\ f^* \\ g^* \end{pmatrix}(e, f, g) = \begin{pmatrix} u & \cdot & \cdot \\ \cdot & w & \cdot \\ \cdot & \cdot & z \end{pmatrix}$$

$E \longrightarrow L(E)$

many other ways to impose these relations

ultra graph / Leavitt path algebra

hyper graph /

separated graph / ⟶

$ee^* + ff^* = u$

$gg^* + hh^* = u$

. . . . .

weighted graph

higher rank graphs

original Leavitt algebra

$$L_n = \frac{\langle x_i, y_i \mid 1 \le i \le n \rangle}{\left\langle (x_1, \dots, x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = id, \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}(x_1, \dots, x_n) = id \right\rangle}$$

but $$L(n,m) = \frac{\langle x_{ij}, y_{ji}, \ \begin{smallmatrix} 1 \le i \le n \\ 1 \le j \le m \end{smallmatrix} \rangle}{\left\langle \begin{pmatrix} x_{ij} \end{pmatrix}\begin{pmatrix} y_{ji} \end{pmatrix} = id, \quad \begin{pmatrix} y_{ji} \end{pmatrix}\begin{pmatrix} x_{ij} \end{pmatrix} = id \right\rangle}$$

<u>Recall</u> $L(n,m)$ is a domain so can't realise as $L(E)$ which has a "lot" of divisors

So weighted graph / weighted Leavitt path algebras.

E a graph, $w : E^1 \to \mathbb{N}$ weight map

$$L(E,w) = \frac{K\langle u, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \le i \le w(e) \rangle}{\begin{array}{c} uv = \delta_{uv} u \quad u, v \in E^0 \\ s(e) e_i = e_i r(e), \quad r(e) e_i^* = e_i^* = e_i^* s(e) \\ \text{where } e \in E^1, \quad 1 \le i \le w(e) \end{array}}$$

③

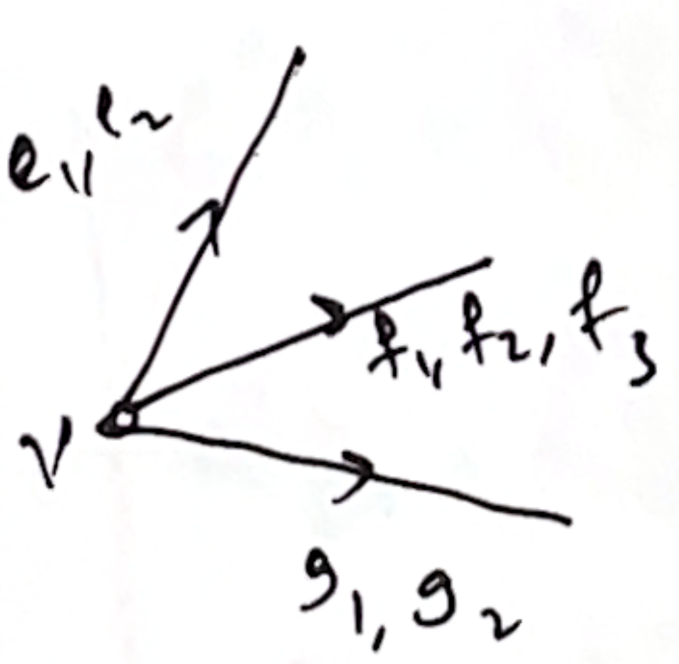
* $$\sum_{e \in s^{-1}(v)} e_i e_j^* = \delta_{ij} v \quad , \quad v \in E^0 \text{ regular}, \quad 1 \le i, j \le w(v)$$

* $$\sum_{1 \le i \le w(v)} e_i^* f_i = \delta_{e,f}\, r(f) \quad , \quad v \in E^0_{\text{regular}}, \quad e, f \in s^{-1}(v)$$

in the last two relation $e_i, e_i^* = 0$ if $i > w(e)$.

$\rightarrow$ $w(v) = \max \{ w(e) \mid e \in s^{-1}(v) \}$

<u>understand</u>



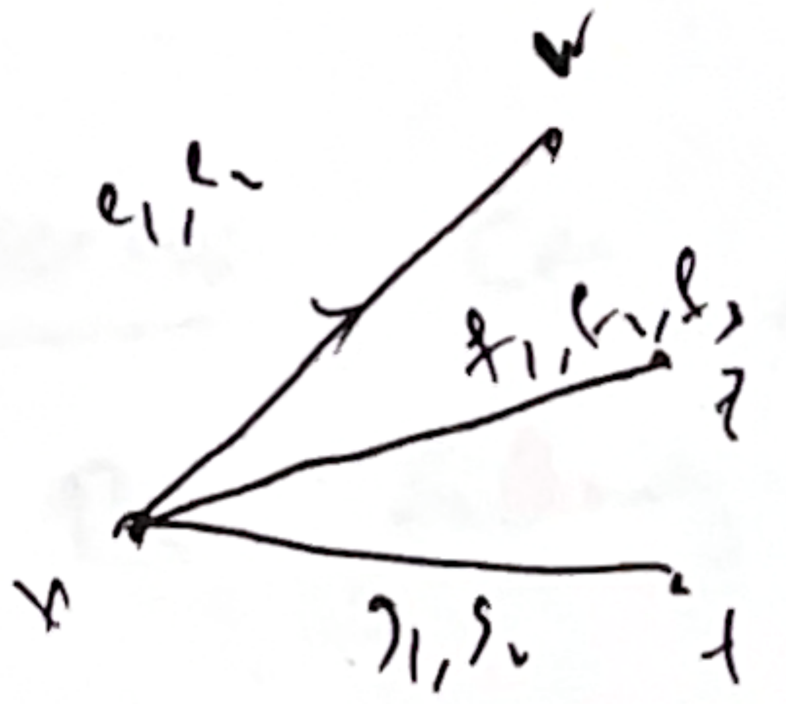
$w(e) = 2$
$w(f) = 3$
$w(g) = 2$

Fix a layer and write CK relations

<u>layer 1</u> $\quad e_1 e_1^* + f_1 f_1^* + g_1 g_1^* = v$

<u>layer 2</u> $\quad e_2 e_2^* + f_2 f_2^* + g_2 g_2^* = v$

<u>layer 3</u> $\quad f_3 f_3^* = v$



for the other relation Fix an edge a go through all layers

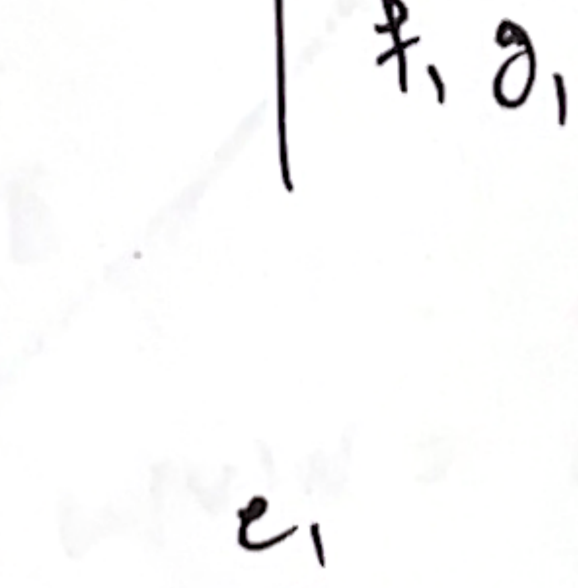
$\underline{e}$

$$e_1^* e_1 + e_2^* e_2 = w$$
$$f_1^* f_1 + f_2^* f_2 + f_3^* f_3 = z$$
$$g_1^* g_1 + g_2^* g_2 = t$$

mix edges

$$e_1^* f_1 + e_2^* f_2 + \underbrace{e_3^* f_3}_{0} = 0$$
$$e_1^* g_1 + e_2^* g_2 = 0$$
$$f_1^* g_1 + f_2^* g_2 = 0$$

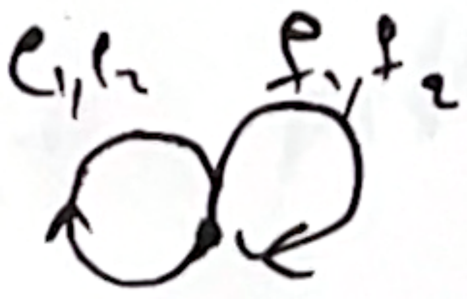
Ex



n-loops of weight

$w(e_i) =$

Exercise: Can you convince yourself that the following algebras are not isomorphic

(E, w) : 

$w(e) = 2$
$w(f) = 2$ $\qquad L(E, w)$

(F, w) : 

$L(F)$

$L(E, w) \ncong L(F)$ $\qquad$ (their monoids are different, one is domain)

Recall the graph monoid

(E, w) weighted ~~graph~~ vertex graph means $w(e) = w(f)$
if $e, f \in s^{-1}(x)$

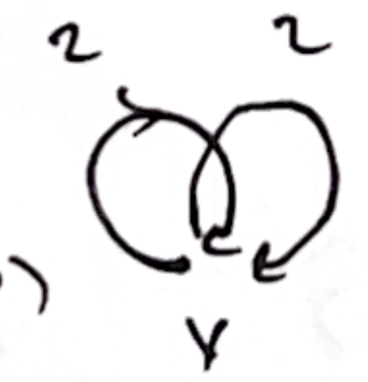
$$M_{(E,w)} = \mathbb{F}_E \Big/ \left\langle w(v)v = \sum_{e \in s^{-1}(v)} r(e) \;\middle|\; v \in E^0_{\text{regular}} \right\rangle$$

if $w = 1$ then $M_{(E,w)} \cong M_E$

Th (E,w) vertex weighted graph $M_{(E,w)}$ graph monoid, and $L(E,w)$ the weighted Leavitt path algebra. Then

$$\mathcal{V}(L_k(E,w)) \cong M_{(E,w)}$$

$$[L_k(E,w)] = \sum_{v \in E^0} v$$

Ex (E,w) 

$$M_{(E,w)} = \frac{\langle v \rangle}{\langle 2v = v+v \rangle} = \mathbb{N}$$

$(F,1)$ 

$$M = \frac{\langle v \rangle}{\langle v = v+v+v+v \rangle} = \{0, v, 2v, 3v\} = C_4$$

so $\mathcal{V}(L_k(E,w)) = \mathbb{N} \implies L(E,w) \not\cong L_k(F)$

$\mathcal{V}(L_k(F)) = C_4$

weighted Lpa not quite understood

Ex

E

$w(e)=2$
$w(f)=1$

$L(E,w)$ is simple

F

$w(e)=2$
$w(f)=w(g)=1$

$L(F)$ is not simple

R. Preusser "weighted Lpas, an overview" 2024

<u>Theorem</u> E sandpile graph, $S \subseteq E$ vertices of E which do not connect to any cycle. Let w be balanced weight in E.

Then $SP(E)/Z(SP(E)) \cong \mathcal{V}(L_K(E/S), w)$

In particular if every vertex not connected to cycle is irrelevant $|s^{-1}(v)|=1$ then

8

$SP(E)$ is conical i.e. $Z(SP(E)) = 0$

So $SP(E) \cong V(L_K(E/S), w)$

$G(E) \cong K_0(L_K(E/S), w)$

Proof We know $SP(E) \cong M(E, w)$

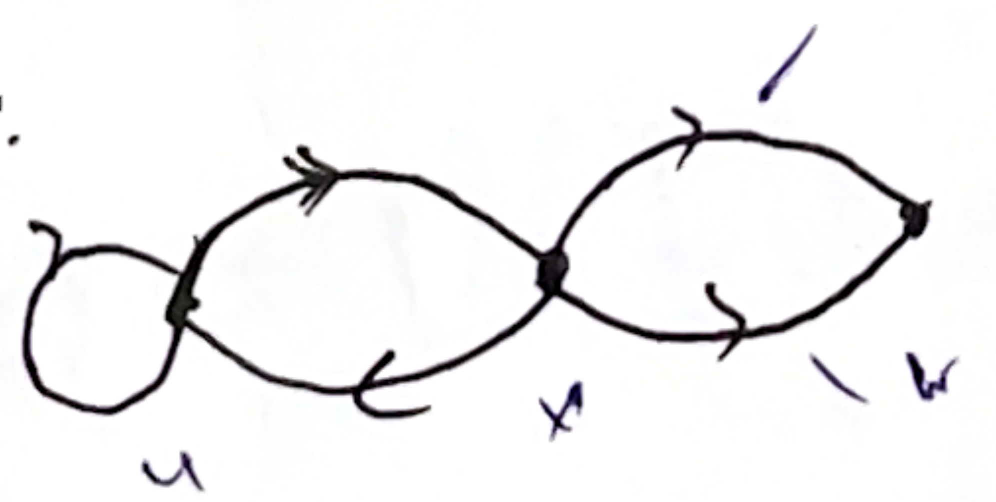
also $SP(E) / Z(SP(E)) \cong M(E, w) / M(S, w)$

$\cong M(E/S, w)$

Since E/S has no sink then

$M(E/S, w) = M_{(E/S, w)}$ so $\cong V(L(E/S, w))$.

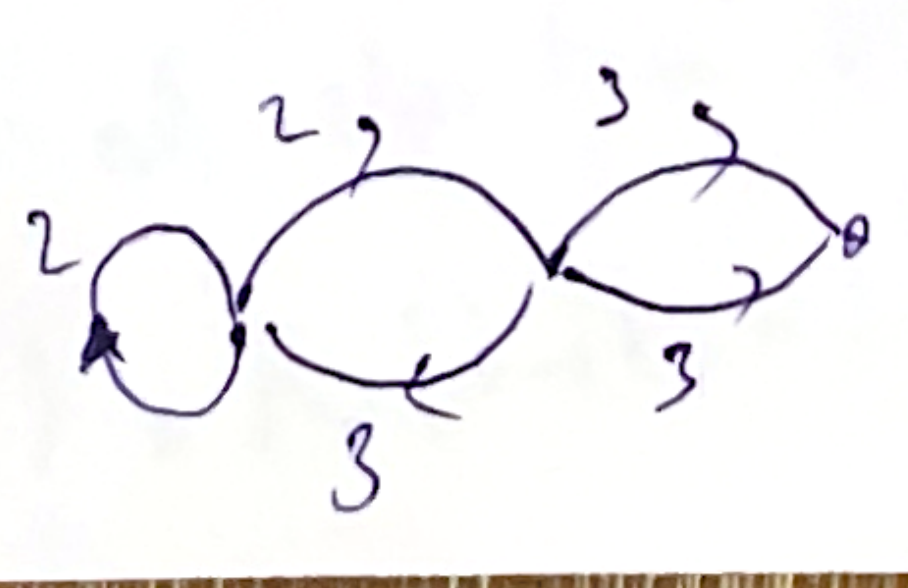
Ex E:



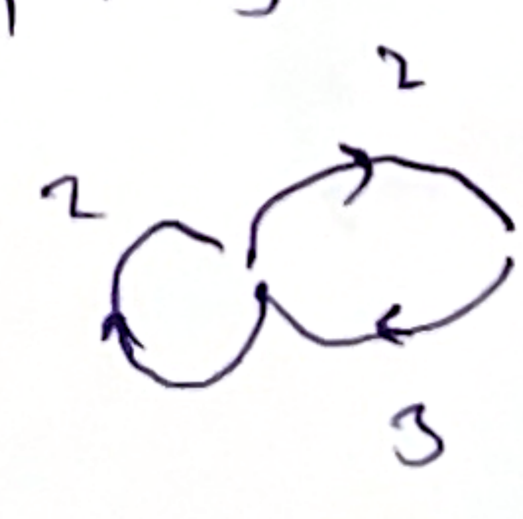
not connected to cycle

$S = \{v, w\}$

(E, w)



E/S :



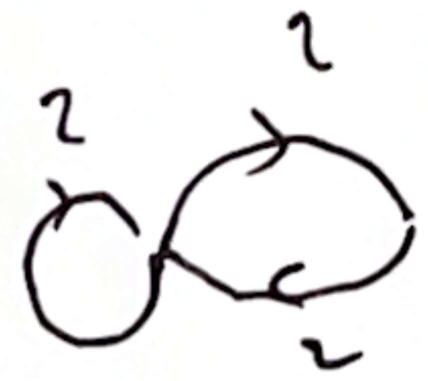
$\emptyset$

$$Sp(E) / Z(Sp(E)) \cong \mathcal{V}(L($$  $$))$$

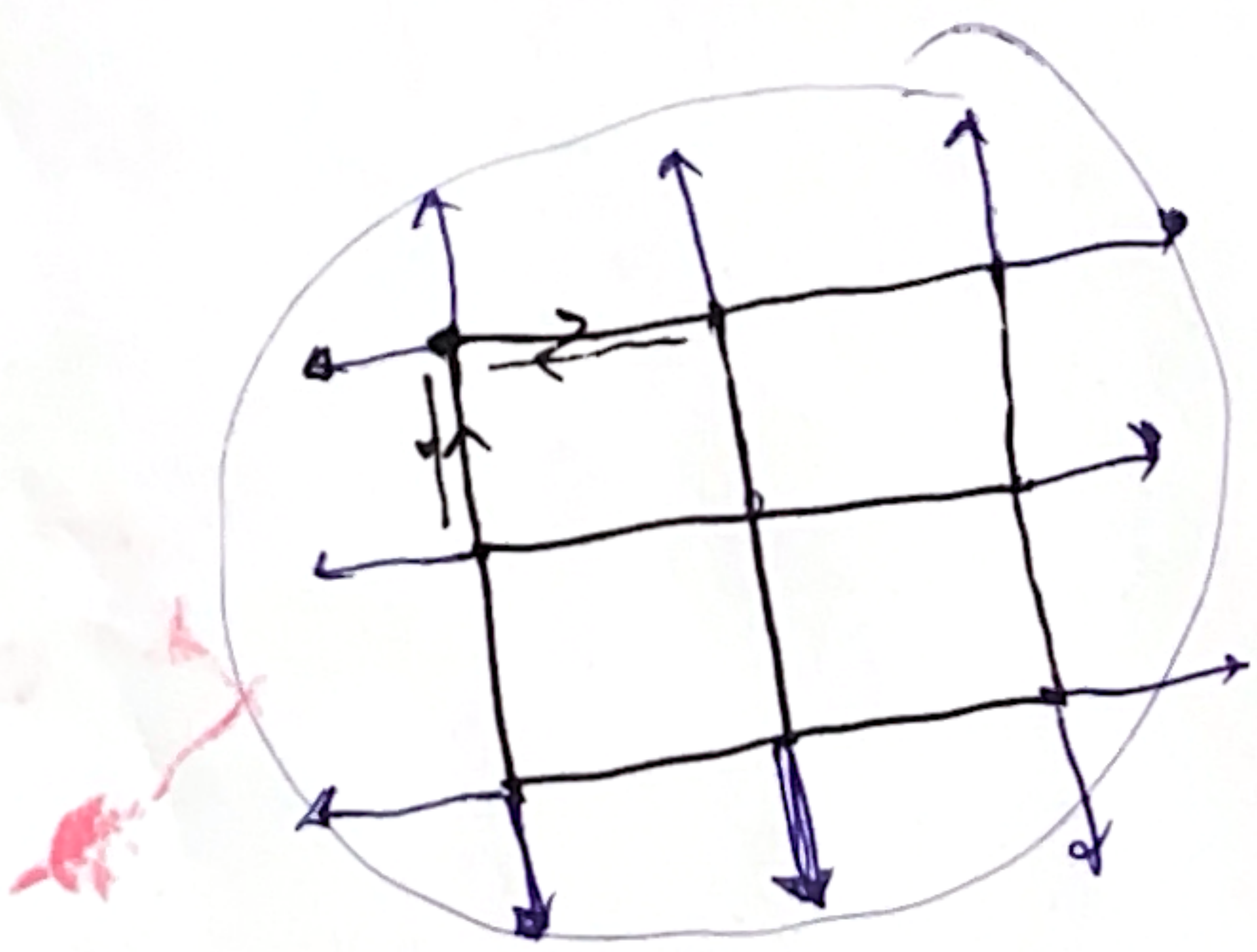
Even nicer

F 

irrelevant

$$Sp(F) \cong \mathcal{V}(L($$  $$)).$$

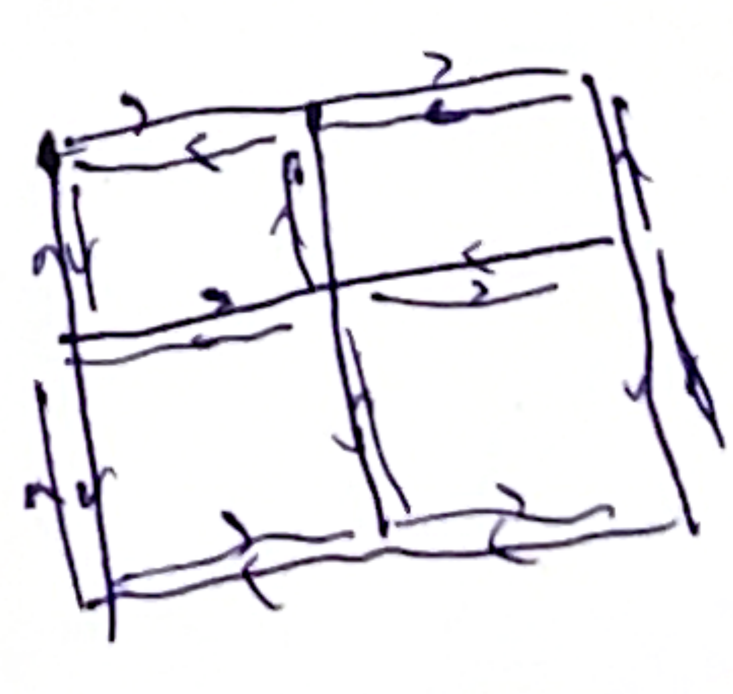
Ex



one sink

S vertices does not connect to a cycle = sink

$$Sp(E) \cong \mathcal{V}(L(E,w))$$ when



all edges have weight 4.

$$G(E) \cong K_0(L(E,w)).$$

10

$$G(E) = H_0(\mathcal{L}(E, w))$$

zero element

zero element

fractal shape

Can this ~~have~~ some interpretation as a standard of $L(E, w)$?